M.Sc. - I (Mathematics) (NEP Pattern) Semester-I

NEP-64-2 / DSE2 Paper-I - Real Analysis

P. Pages : 2 **GUG/S/25/15116**

Time : Three Hours Max. Marks : 80

Notes : 1. Solve all the **five** questions.
2. All questions carry equal marks.

UNIT - I

1. a) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions on a set E. Then prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on E if and only if for every $\in > 0$, there exist an integer N such that $m \ge N, n \ge N, x \in E \Rightarrow |f_m(x) - f_n(x)| \le \in$. **8**

b) If $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions R[a, b] and if $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on [a, b] then prove that f is also in R [a, b]. **8**

OR

c) Let $\sum_{k=1}^{\infty} u_k$ be a series of functions in R [a, b] that converges uniformly to f on [a, b]. Then prove that $f \in R[a, b]$ and $\int_a^b f(x)dx = \sum_{k=1}^{\infty} \int_a^b u_k(x)$. **8**

d) Prove that there exist a real continuous function on the real line which is no where differentiable. **8**

UNIT - II

2. a) Let A be an open in R^m. Suppose that the partial derivatives $D_j f_i(x)$ of the component functions of f exist at each point x of A and are continuous on A. Then prove that f is differentiable at each point of A. **8**

b) Let A be open in R^m; Let $f : A \to R$ be a function of class C^2. Then prove that for each $a \in A, D_k D_j f(a) = D_j D_k f(a)$. **8**

OR

c) Let A be open in R^n; let $f : A \to R^n$ be class C^1. If Df(a) is non-singular then prove that there exists an $\alpha > 0$ such that inequality **8**

$$|f(x_0) - f(x_1)| \ge \alpha |x_0 - x_1|,$$

holds for all x_0, x_1 in some cube $C(a; \in)$ centred at a.

d) State and prove the implicit function theorem. **8**

UNIT - III

3. a) Let Q be rectangle; $f: Q \to R$ be a bounded function. Then prove that $\underline{\int_Q} f \le \overline{\int_Q} f$ and equality holds if and only if given $\in > 0$, there exists a corresponding partition P of Q for which $U(f,P) - L(f,P) < \in$. **8**

b) Let Q be a rectangle in R^n; let $\{Q_1, \ldots, Q_k\}$ be a finite collection of rectangles that covers Q. then prove that $v(Q) \le \sum_{i=1}^{k} v(Q_i)$. **8**

OR

c) State and prove Fubini's theorem. **8**

d) Let A be a bounded open set in R^m; let $f: A \to R$ be a bounded continuous function. Then the extended integral $\int_A f$ exists. If the ordinary integral $\int_A f$ also exists then prove that these two integrals are equal. **8**

UNIT - IV

4. a) Let A be open in R^n; let $f: A \to R$ be continuous. Let $\{\phi_i\}$ be a partition of unity on A having compact supports. Then prove that the integral $\int_A f$ exists if and only if the series $\sum_{i=1}^{\infty}\left[\int_A \phi_i |f|\right]$ converges. **8**

b) State and prove change of variables theorem. **8**

OR

c) Let $h: R^n \to R^n$ be a map such that h(0) = 0. Then prove that: **8**
i) The map h is an isometry if and only if it preserves dot products.
ii) The map h is an isometry if and only if it is an orthogonal transformation.

d) Let $g: A \to B$ be a diffeomorphism of class C^r, where A and B are open sets in R^n. Let D be a compact subset of A and let E = g (D) then prov that: **8**
i) g(Int D) = Int E and g (Bd D) = Bd E
ii) If D is rectifiable, so is E.

5. a) Define pointwise convergence of series of functions and uniform convergence of sequence of function. **4**

b) Let $A \subset R^m$; let $f: A \to R^n$. If f is differentiable at a then prove that if is continuous at a. **4**

c) Write fundamental theorem of calculus. **4**

d) Write statement of existence of a partition of unity. **4**

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